ON THE BAND-SPECTRAL ESTIMATION OF BUSINESS CYCLE MODELS

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  - exact likelihood (time domain) estimator

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- Monte Carlo study:
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    - full-spectrum (Whittle) estimator
    - exact likelihood (time domain) estimator
- assess approximation distortions and information loss

Why band-spectral estimation?

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- estimate models which are a priori known to be unable to represent some frequencies

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"***Quantifying confidence***" by Angeletos, Collard, and Dellas (2018)

The model described above – like other business-cycle models – cater to ***business-cycle phenomena*** *and therefore omit shocks and mechanisms that may account for medium- to long-run phenomena, such as trends in demographics and labor-market participation, structural transformation, regime changes in productivity growth or inflation, and so on.*

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- ACD estimate their model with BSE using BC frequencies only

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   Note: full info, i.e. when misspecification is not a concern.

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    - we can reliably predict the loss of efficiency

- Likelihood
- Monte Carlo: setup, results
- Conclusion

Gaussian Likelihood Function

Gaussian Likelihood Function

Let $\boldsymbol{y}_t$ be a stationary Gaussian process with zero mean.

$$\boldsymbol{Y}_T = \left(\boldsymbol{y}_1', \boldsymbol{y}_2', \ldots, \boldsymbol{y}_T'\right)' \sim \mathcal{N}\left(\mathbf{0}, \boldsymbol{\Sigma}_T(\boldsymbol{\theta})\right)$$

The log-likelihood is

$$\ell(\boldsymbol{\theta}; \boldsymbol{Y}_T) = -\frac{1}{2}\log\det(\boldsymbol{\Sigma}_T(\boldsymbol{\theta})) - \frac{1}{2}\boldsymbol{Y}_T'\boldsymbol{\Sigma}_T^{-1}(\boldsymbol{\theta})\boldsymbol{Y}_T \quad (1)$$

Gaussian Likelihood Function

Whittle approximation: replace $\boldsymbol{\Sigma}_T(\boldsymbol{\theta}) \approx \boldsymbol{\Omega}_T(\boldsymbol{\theta}) = \boldsymbol{F}_T^* \boldsymbol{S}_T(\boldsymbol{\theta}) \boldsymbol{F}_T$

$$\ell(\boldsymbol{\theta}; \boldsymbol{Y}_T) = -\frac{1}{2} \log\det(\boldsymbol{\Sigma}_T(\boldsymbol{\theta})) - \frac{1}{2} \boldsymbol{Y}_T' \boldsymbol{\Sigma}_T^{-1}(\boldsymbol{\theta}) \boldsymbol{Y}_T \quad (1)$$

$$\approx -\frac{1}{2} \log\det(\boldsymbol{S}_T(\boldsymbol{\theta})) - \frac{1}{2} (\boldsymbol{F}_T \boldsymbol{Y}_T)^* \boldsymbol{S}^{-1}(\boldsymbol{\theta}) (\boldsymbol{F}_T \boldsymbol{Y}_T) \quad (2)$$

$$\approx -\frac{1}{2} \sum_{\omega=\omega_1}^{\omega_T} \left\{ \log\det(\boldsymbol{s}(\boldsymbol{\theta}, \omega)) + \tilde{\boldsymbol{y}}(\omega)^* \boldsymbol{s}^{-1}(\boldsymbol{\theta}, \omega) \tilde{\boldsymbol{y}}(\omega) \right\} \quad (3)$$

- $\boldsymbol{F}_T$ is Fourier transform matrix
- $\boldsymbol{S}_T(\boldsymbol{\theta})$ is block-diagonal

Three estimators

- **TD** maximizes (full info, KF)

$$\ell(\boldsymbol{\theta}; \boldsymbol{Y}_T) = -\frac{1}{2}\log\det(\boldsymbol{\Sigma}_T(\boldsymbol{\theta})) - \frac{1}{2}\boldsymbol{Y}_T'\boldsymbol{\Sigma}_T^{-1}(\boldsymbol{\theta})\boldsymbol{Y}_T$$

- **FD** maximizes (full info, Whittle, all freqs)

$$\ell_w(\boldsymbol{\theta}; \boldsymbol{I}_T) = -\frac{1}{2}\sum_{all\ \omega}\left\{\log\det(\boldsymbol{s}(\boldsymbol{\theta},\omega)) + \tilde{\boldsymbol{y}}(\omega)^*\boldsymbol{s}^{-1}(\boldsymbol{\theta},\omega)\tilde{\boldsymbol{y}}(\omega)\right\}$$

- **BC** maximizes (limited info, Whittle, BC freqs - periodicity between 6 and 32 quarters)

$$\ell_w(\boldsymbol{\theta}; \boldsymbol{I}_T^{BC}) = -\frac{1}{2}\sum_{\omega\in\bar{\boldsymbol{\omega}}^{BC}}\left\{\log\det(\boldsymbol{s}(\boldsymbol{\theta},\omega)) + \tilde{\boldsymbol{y}}(\omega)^*\boldsymbol{s}^{-1}(\boldsymbol{\theta},\omega)\tilde{\boldsymbol{y}}(\omega)\right\}$$

MONTE CARLO

DGP

New Keynesian DSGE model from “Quantifying confidence” by Angeletos, Collard, and Dellas (2018)

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- the confidence shock represents perceived bias in the other agents' expectations about the level of TFP in each period (higher-order beliefs)
  - leads to waves of optimism (believing that others are optimistic) and pessimism (believing that others are pessimistic) that generate business cycle fluctuations unrelated to fundamentals

DGP

New Keynesian DSGE model from “Quantifying confidence” by Angeletos, Collard, and Dellas (2018)

- 25 estimated parameters
- six observed variables: GDP, consumption, investment, hours worked, inflation, and the federal funds rate
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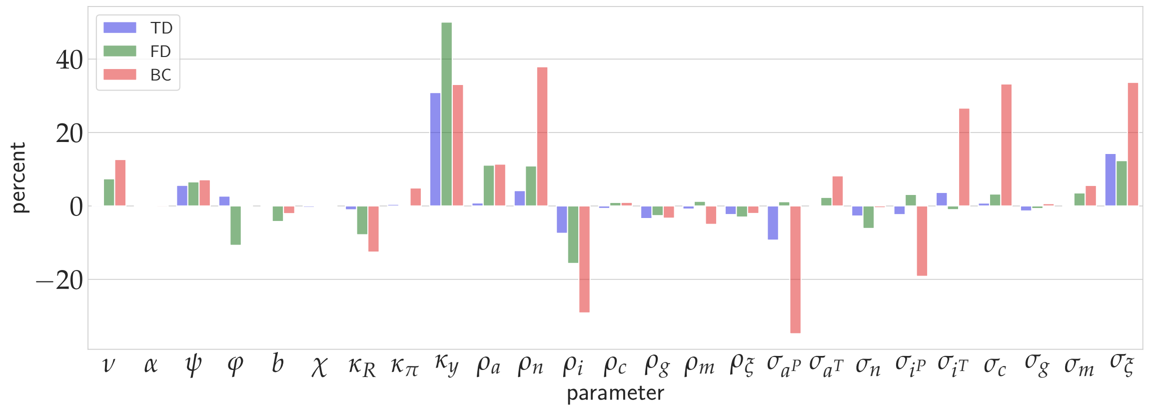
true DGP for **all** frequencies

RESULTS

1000 replications

Bias (% of θ)



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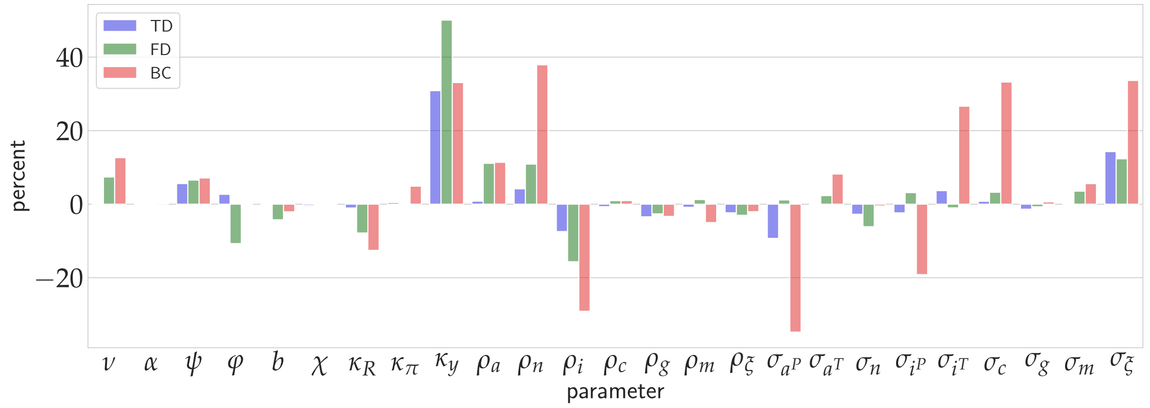
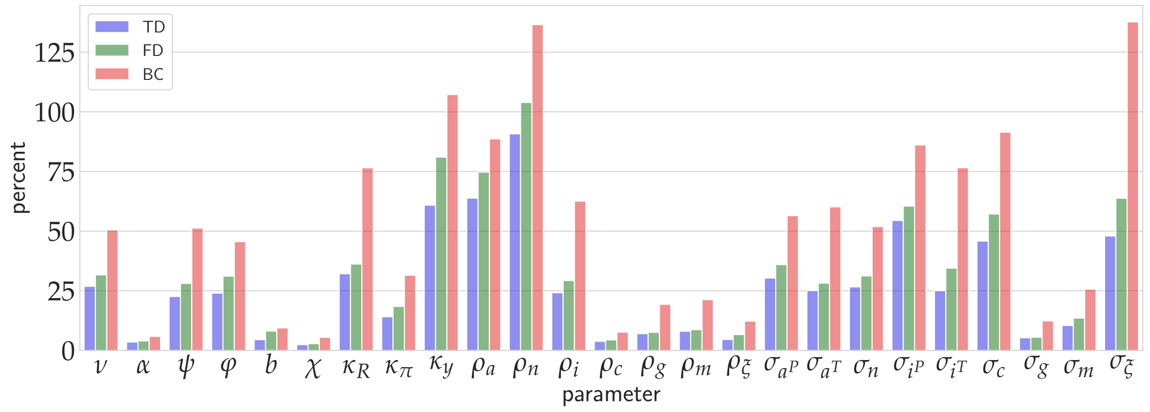
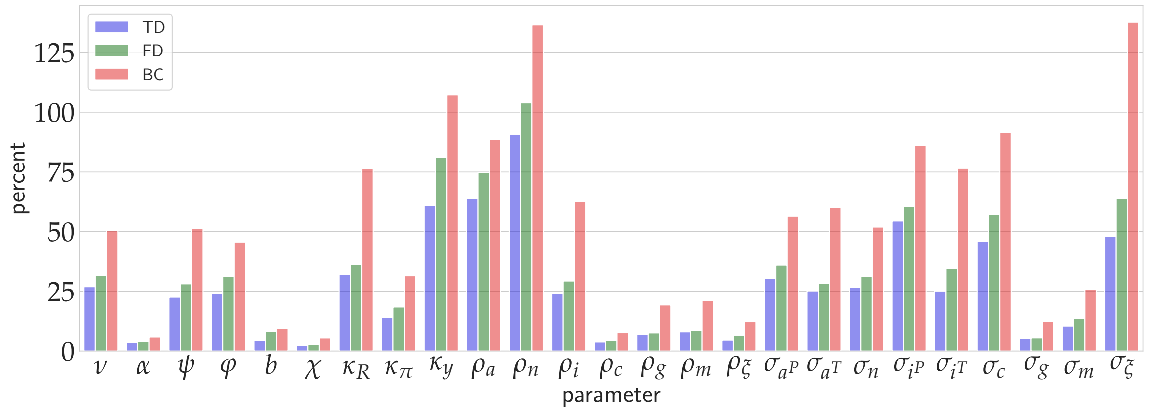


- 4%, 7%, 13%
- rank corr: 0.57 (TD, FD), 0.84 (TD, BC), 0.66 (FD, BC)

Efficiency (std as % of θ)



Efficiency (std as $\%$ of θ)



- 27%, 32%, 53%
- rank corr: 0.98 (TD, FD), 0.95 (TD, BC)

Can the efficiency loss be predicted?

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- Cramér-Rao lower bound (CRLB): if $\hat{\boldsymbol{\theta}}$ is unbiased, then

$$\mathrm{std}_{\hat{\theta}_i} \geq \sqrt{\{\mathcal{I}^{-1}(\boldsymbol{\theta})\}_{ii}} = \mathrm{crlb}_{\hat{\theta}_i} \tag{4}$$

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- predicted efficiency loss

$$\frac{\mathrm{crlb}(BC)}{\mathrm{crlb}(TD)} \tag{5}$$

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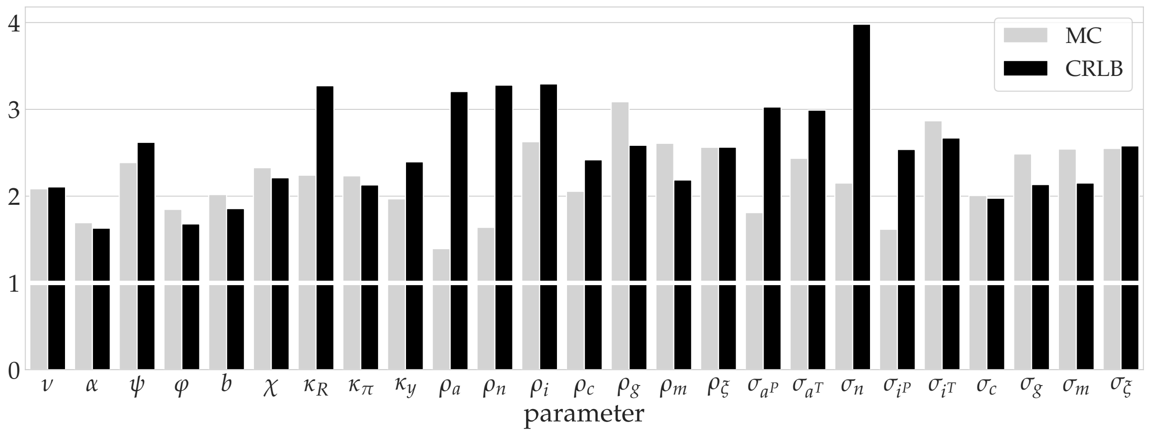
- is

$$\frac{\mathrm{crlb}(BC)}{\mathrm{crlb}(TD)} \approx \frac{\mathrm{std}(BC)}{\mathrm{std}(TD)} \quad ?$$

Efficiency loss

How much less information in the BC frequencies?



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- estimated efficiency loss:

$$\frac{\widehat{\mathrm{std}}(BC)}{\widehat{\mathrm{std}}(TD)} = \frac{88.5}{63.5} = 1.4$$

- predicted efficiency loss:

$$\frac{\mathrm{crlb}(BC)}{\mathrm{crlb}(TD)} = \frac{229}{71.4} = 3.2$$

What's wrong with ρ_a (and ρ_n, κ_R, σ_n, ...)?

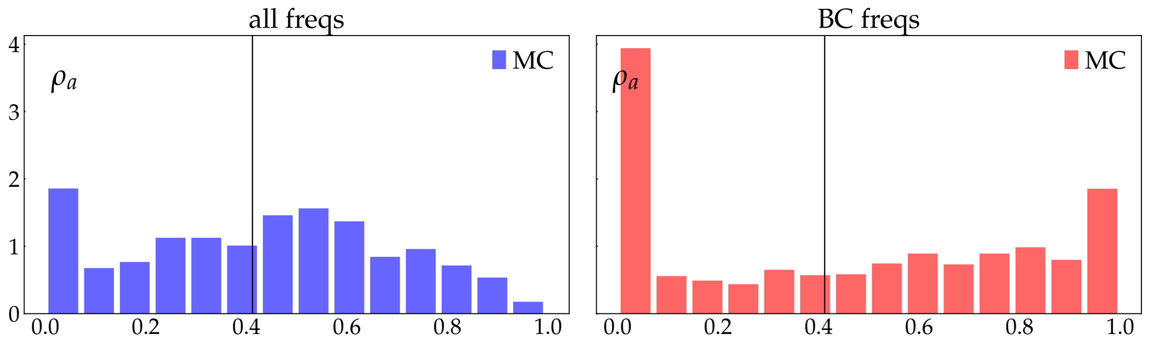
short answer:

- MC overestimates sample information, esp. in BC band.
- Thus, MC underestimates the efficiency loss (loss of sample info).

$$\frac{\mathrm{std}(BC) << \mathrm{crlb}(BC)}{\mathrm{std}(TD) < \mathrm{crlb}(TD)}$$

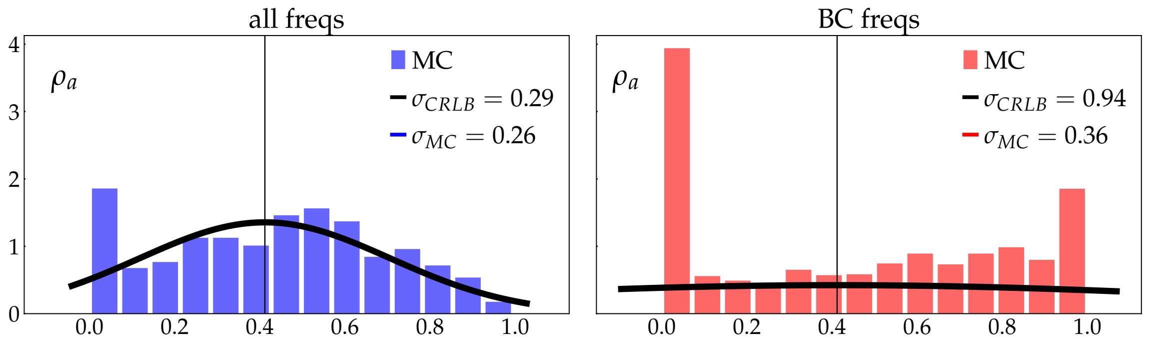
What's wrong with ρ_a (and ρ_n, κ_R, σ_n, ...)?

MC-estimated marginal distribution of $\hat{\rho}_a$



What's wrong with ρ_a (and ρ_n, κ_R, σ_n, ...)?

MC-estimated vs CRLB-predicted marginal distribution of $\hat{\rho}_a$



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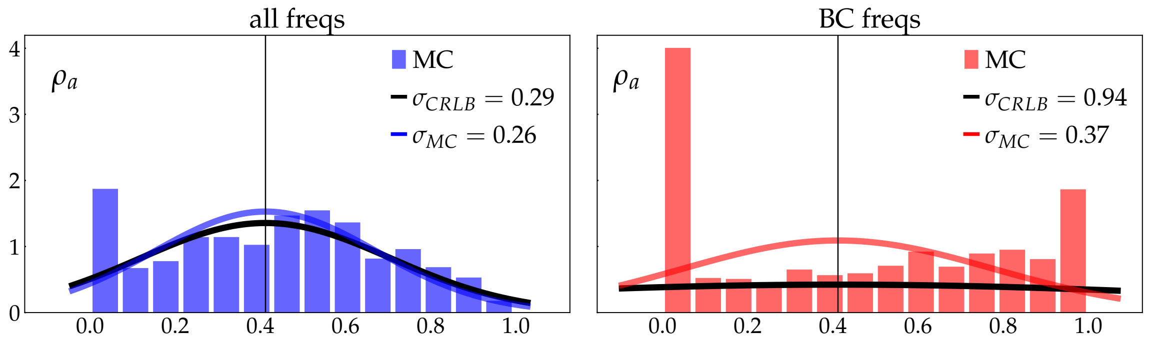
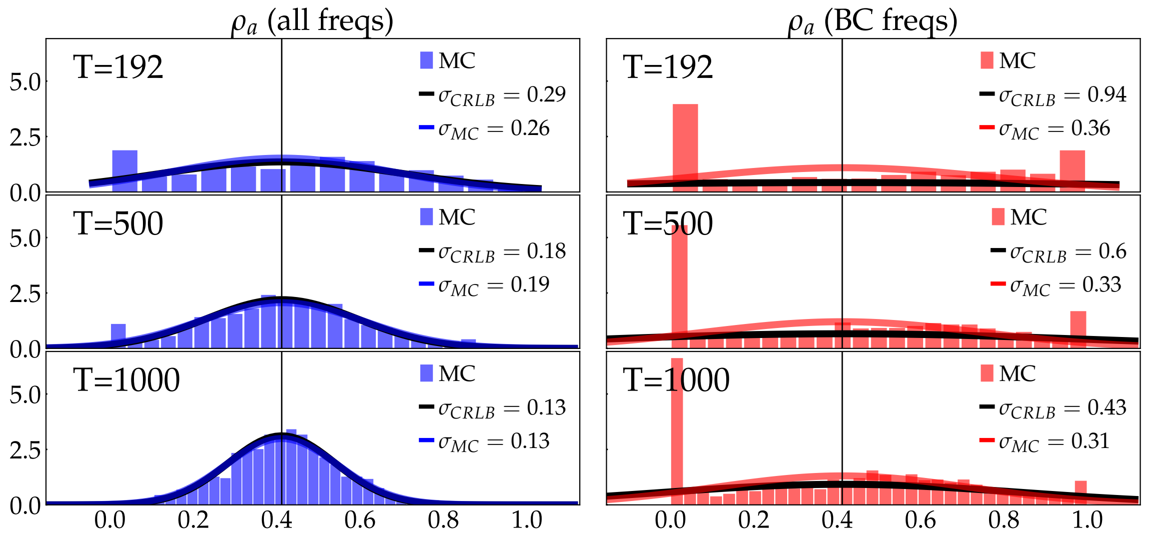


MC underestimates uncertainty (overestimates information **contained in the sample**) due to

- **flat likelihood** (in-sample information deficiency)
- **parameter constraints** (out-of-sample parameter information)

ρ_a: MC vs CRLB as T increases



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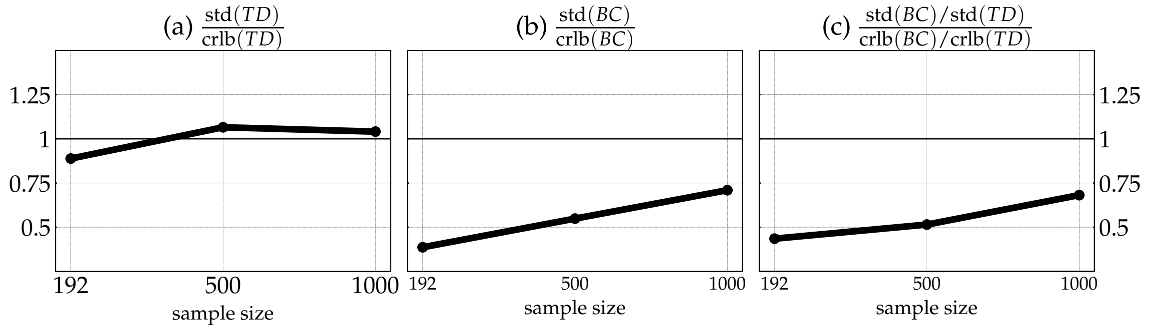

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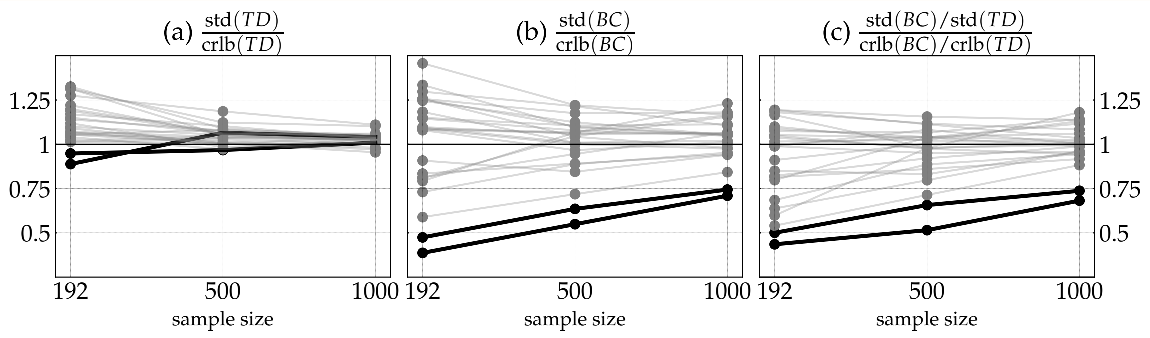

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   *The model described above – **like other business-cycle models** – cater to business-cycle phenomena ... (ACD, 2018)*

   99.9% of BC models are estimated in the time-domain ...

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3. transparency: where does information about θ_i come from?

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   - model property
   - which frequencies are most informative and why

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- band-spectral estimation significantly less efficient
  - a lot of info outside the BC freqs for all parameters
- FIM analysis is useful to assess the loss of information in band-spectral estimation
  - (relative) CRLBs accurately predict (relative) estimation uncertainty

Some Implications

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- Evidence for misspecification
    - estimating a model over different frequency bands leads to different estimates (Qu and Tkachenko (2012), Sala (2015))

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- Evidence for misspecification
  - estimating a model over different frequency bands leads to different estimates (Qu and Tkachenko (2012), Sala (2015))
  - might be true even if the model is **not** misspecified
    - bias
    - information deficiencies

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- Calibration
  - weakly identified parameters are often calibrated

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  - weakly identified parameters are often calibrated
  - may have to calibrate (many) other parameters for band-spectral estimation

Some Implications

- Bayesian estimation and importance of priors

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- Bayesian estimation and importance of priors
    - the same prior is much more informative with band-spectral estimation

APPENDIX

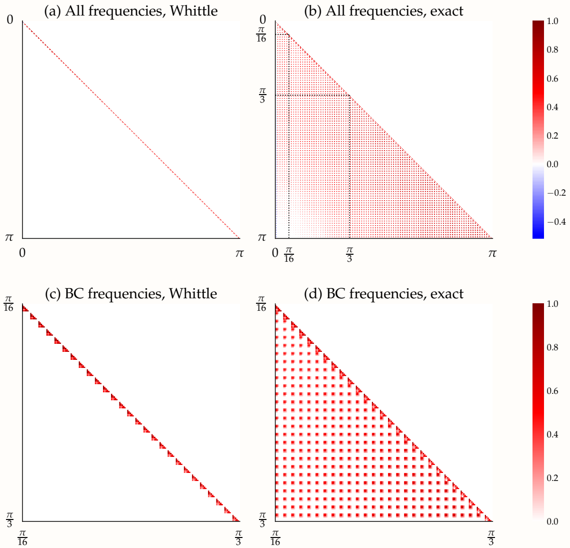
Sala (2015)

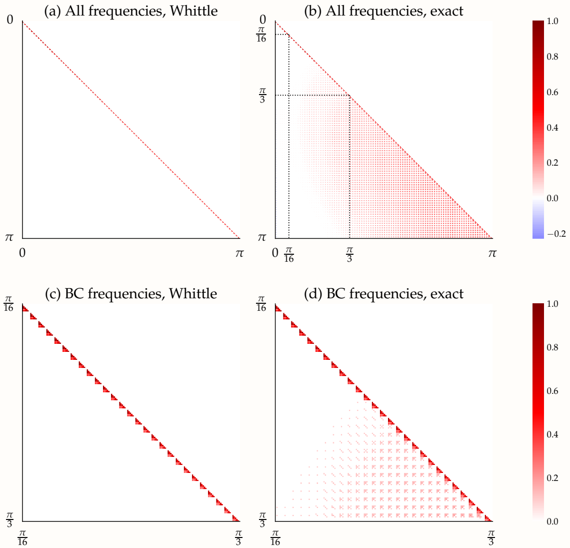
- Monte Carlo experiment with NK DSGE model
- 100 samples $T = 170$
- KF, All, Low-Pass, High-Pass, BC
- "*In sum, the evidence shows that, when using the DSGE model as data-generating process, maximum likelihood in the frequency domain is **equivalent** to maximum likelihood in the time domain, and that the precision of the estimates is still **very good** when estimation is performed on frequency bands*"

$$\boldsymbol{A} = \begin{bmatrix} A_0 & A_1 & \cdots & A_1' \\ A_1' & A_0 & \cdots & A_2' \\ \vdots & \vdots & \ddots & \vdots \\ A_1 & A_2 & \cdots & A_0 \end{bmatrix},$$

Table: posterior median

ψ	utilization elasticity	0.500
ν	inverse labor supply elasticity	0.282
α	capital share	0.255
φ	investment adjustment costs	3.312
b	habit persistence	0.758
χ	Calvo parameter,	0.732
κ_R	Taylor rule smoothing,	0.198
κ_π	Taylor rule inflation,	2.271
κ_y	Taylor rule output,	0.121
ρ_m	AR mon. policy	0.647
ρ_a	AR transitory TFP component	0.412
ρ_n	AR news	0.224
ρ_i	AR transitory investment-specific technology	0.374
ρ_c	AR preference	0.888
ρ_g	AR government spending	0.786
ρ_ξ	AR confidence	0.833
σ_a^P	std. permanent TFP component	0.406
σ_a^T	std. transitory TFP component	0.347
σ_n	std. news	0.378
σ_i^P	std. permanent investment-specific technology	0.610
σ_i^T	std. transitory investment-specific shocks	5.805
σ_c	std. preference	0.357
σ_g	std. government spending	1.705
σ_ξ	std. confidence	0.613
σ_m	std. mon. policy	0.313


(a) All frequencies, Whittle
(b) All frequencies, exact
(c) BC frequencies, Whittle
(d) BC frequencies, exact
0
π
$\frac{\pi}{16}$
$\frac{\pi}{3}$
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4



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π
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π/3
1.0
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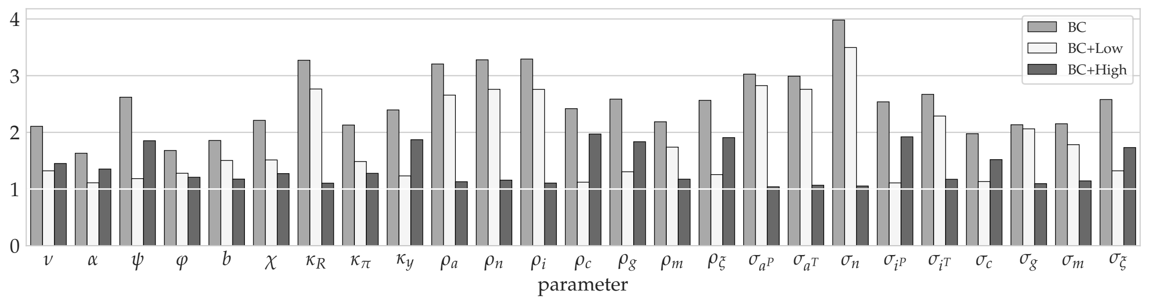

4
3
2
1
0
BC
BC+Low
BC+High
ν α ψ φ b χ κ_R κ_π κ_y ρ_a ρ_n ρ_i ρ_c ρ_g ρ_m ρ_ξ σ_{a^P} σ_{a^T} σ_n σ_{i^P} σ_{i^T} σ_c σ_g σ_m σ_ξ
parameter